Lecture IV

→ "Big Picture" of Equilibrium { Flctns Dynamics → ©

Topic I → { Thermal Equilibrium Plasma

→ Thermal Equilibrium Plasma : **Basic Ideas**

- simplest possible dynamics question

⇒ What is spectrum of thermal equilibrium fluctuations in plasma?

- answer → determined by balance between

→ emission and absorption

i.e.

→ Fluctuation ↔ Dissipation

What is key physics of each?

} ⇒ Physics of F.D.T.

Generic Consideration

Consider some simple examples:

simplest

- particle undergoing Brownian force in fluid → see (a)

particle in fluid at temp T i.e.

$$m\frac{d\underline{v}}{dt} = -\gamma m\underline{v} + \underline{f}$$

Stokes drag $\sim v$

thermal fluctuations

- $\tilde{f}$ → random (statistical), so uncorrelated in time

→ correlation time

$$\langle \tilde{f}(t_1)\,\tilde{f}(t_2)\rangle = 2\tilde{f}_0^2\,\tau_c\,\delta(t_1-t_2)$$

auto-correlation function.

Note:

- standard notation for stokes drag is:

mass of Brownian particle

$$m \frac{d\underline{v}}{dt} = -\gamma \underline{v} + \underset{\sim}{\underline{f}}$$

$$\gamma = 6\pi\eta a$$

$$\eta = \rho\nu$$

fluid mass density

So, in these notes:

$\gamma \to \gamma/m$, as write;

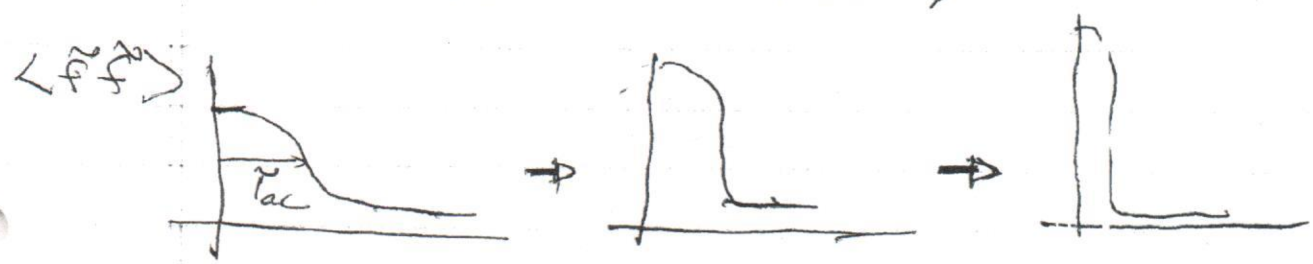
$$\left[m \frac{d\underline{v}}{dt} = -m(\gamma/m)\underline{v} + \tilde{\underline{f}} \right.$$

What is τ_c ? → spectral auto-correlation time (self-coherence)

→ measures self-correlation of $\tilde{\ }$ random force.

i.e. if stationary,

$$\langle \tilde{f}(0)\,\tilde{f}(\tau)\rangle = \langle \tilde{f}(t_1)\,\tilde{f}(t_2)\rangle$$



⇒ for "white noise"
$\tau \ll$ all other time scales

now,

$$\frac{d\tilde{v}}{dt} + \gamma\,\tilde{v} = \frac{\tilde{f}}{m}$$

$$\tilde{v}(t) = e^{-\gamma t}\,\tilde{v}(0) + \int_0^t dt'\, e^{-\gamma(t-t')}\,\frac{\tilde{f}}{m}(t')$$

{ no correlation i.e. with $\tilde{f}$

Now $$|\tilde{v}|^2 = e^{-2\gamma t}\,|\tilde{v}(0)|^2 + \cancel{\langle \tilde{v}\tilde{f}\rangle}$$
$$+ \int^t dt'\, e^{-\gamma(t-t')}\,\frac{\tilde{f}}{m}(t') \int^t dt''\, e^{-\gamma(t-t'')}\,\frac{\tilde{f}}{m}(t'')$$

cross terms

$$|\tilde{v}|^2 = e^{-2\gamma t}|\tilde{v}(0)|^2 + \int_0^t dt' \int_0^t dt'' \, e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \frac{\tilde{f}(t')\tilde{f}(t'')}{m^2}$$

$$\langle |\tilde{v}|^2 \rangle = e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \int_0^t dt' \int_0^t dt'' \, e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \frac{\langle \tilde{f}(t')\tilde{f}(t'')\rangle}{m^2}$$

ensemble statistical average

(2 for symmetrization)

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \int_0^t dt' \int_0^t dt'' \, e^{-\gamma(t-t'')} e^{-\gamma(t-t')} \frac{2\tilde{f}_0^2 \tau_c}{m^2} \delta(t'-t'')$$

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \int_0^t dt' \, e^{-2\gamma(t-t')} \frac{2\tilde{f}_0^2 \tau_c}{m^2}$$

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + e^{-2\gamma t} \frac{2\tilde{f}_0^2 \tau_c}{m^2} \frac{1}{2\gamma}\left(e^{2\gamma t} - 1\right)$$

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \frac{\cancel{2}\tilde{f}_0^2 \tau_c}{\cancel{2}\gamma m^2}\left(1 - e^{-2\gamma t}\right)$$

so for t large ($\gamma t >> 1$)

$$\boxed{\langle \tilde{V}^2 \rangle \cong \frac{\tilde{f}_0^2 \tau_c}{\gamma m^2}}$$

but $\quad m \dfrac{\langle \tilde{V}^2 \rangle}{2} \equiv T \quad \rightarrow$ bath of T!

$$\Rightarrow \qquad T \cong \frac{\tilde{f}_0^2 \tau_c}{2 \gamma m}$$

simple

$$\boxed{\frac{\tilde{f}_0^2 \tau_c}{m^2} = \gamma \frac{T}{m}}$$

$\rightarrow$ Fluctuation - dissipation theorem

($\gamma \rightarrow \gamma/m$, m's cancel)

i.e. $\rightarrow$ given
- noise $(\tilde{f}_0^2 \tau_c)$
- damping (γ)
- temperature (T)

must have:

$$\boxed{(\text{noise}) = (\text{damping})\, T}$$

$\rightarrow$ given 2 of 3 $\Rightarrow$ deduce third!

$$\frac{\partial}{\partial t}\tilde{v} + \gamma\tilde{v} = \frac{\tilde{f}}{m}$$

$\tilde{v}\,\times$

stationarity

$$\cancel{\frac{\partial}{\partial t}\frac{\langle\tilde{v}^2\rangle}{2}} + \gamma\langle\tilde{v}^2\rangle = \frac{\langle\tilde{f}\tilde{v}\rangle}{m}$$

but

i.c.

$$\tilde{v}(t) = \cancel{e^{-\gamma t}\tilde{v}(0)} + \int_0^t dt'\, e^{-\gamma(t-t')}\frac{\tilde{f}(t')}{m}$$

$$\langle\tilde{v}^2\rangle = \frac{T}{m} = \frac{1}{\gamma}\left\langle\tilde{f}\int_0^t dt'\, e^{-\gamma(t-t')}\frac{\tilde{f}(t')}{m^2}\right\rangle$$

$$\langle\tilde{f}(t)\tilde{f}(t')\rangle = |\tilde{f}|^2\tau_c\,\delta(t-t')$$

$$\langle\tilde{v}^2\rangle = \frac{T}{m} = \left(\frac{1}{\gamma}\right)\frac{|\tilde{f}|^2\tau_c}{m^2}$$

$$\boxed{\frac{\langle\tilde{f}^2\rangle}{m}\tau_c = \gamma T}$$

$$\frac{\tilde{f}_0^2 \cdot \gamma_0}{M} = \gamma T$$

but $m\gamma \to \gamma'$ (usual)

$\therefore$

$$\frac{\tilde{f}_0^2 \gamma_0}{M} = \frac{\gamma'}{m} T$$

$$\boxed{f_0^2 T_0 = \gamma' T}$$ $\leadsto$ standard form.

→ equilibrium:

→ emission by noise

→ absorption by damping

$$\boxed{\begin{array}{l}\Rightarrow \text{balance matches } T \,! \\ T \oplus \text{damping} \rightarrow \text{noise}\end{array}}$$

note: alternatively

$$(-i\omega + \gamma)\tilde{V}_\omega = \tilde{f}_\omega / m$$

$$|\tilde{V}_\omega|^2 = \frac{\left|\frac{\tilde{f}_\omega}{m}\right|^2}{(\omega^2+\gamma^2)}$$

white noise: spectral intensity flat

$$\int d\omega\, |\tilde{V}_\omega|^2 = \frac{2T}{m} = \frac{|\tilde{f}|^2}{m^2}\int \frac{d\omega}{\omega^2+\gamma^2}$$

$$= \frac{|\tilde{f}_\omega|^2/m^2}{\gamma}$$

$$\frac{|\tilde{f}_\omega|^2}{m^2} = 2\gamma \frac{T}{m}$$

→ same

→ factors ⟺ normalization

note

$$\underbrace{|\tilde{V}_\omega|^2}_{\text{response spectral density}} = \frac{\overbrace{|\tilde{f}_\omega|^2/m^2}^{\text{noise spectral density}}}{\underbrace{\omega^2+\gamma^2}_{\text{damping}}}$$

$$= \frac{|\tilde{f}_\omega|^2/m^2}{\underbrace{|r(\omega)|^2}_{\text{response function}}}$$

[damping ⟺ width]

i.e. $\frac{T}{m} = \int \frac{|\tilde{q}|^2}{|r(\omega)|^2} d\omega$

if oscillator:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{\tilde{f}}{m}$$

$$|\tilde{x}_\omega|^2 = \frac{|\tilde{f}_\omega|^2/m^2}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}$$

note: $T/m = \int \frac{|\tilde{a}|^2}{|r_r(\omega)|^2 + |r_{IM}(\omega)|^2} d\omega$

if $\to |\tilde{a}(\omega)|^2$ broad
$\to r(\omega)$ has lines, so

$$r_r(\omega) = (\omega - \omega_0) \frac{\partial r}{\partial \omega}$$

$$T/M = \int \frac{|\tilde{a}|^2 \, d\omega}{(\omega-\omega_0)^2 \left(\frac{\partial r}{\partial \omega}\right)^2 + |r_{IM}(\omega_0)|^2}$$

$$= |\tilde{a}(\omega_0)|^2 \int \frac{d\omega / |r_{IM}(\omega_0)|^2}{\left[\frac{(\omega-\omega_0)^2}{|r_{IM}(\omega_0)|^2} \left|\frac{\partial r_r}{\partial \omega}\right|^2 + 1\right]}$$

$$\cong \frac{|\tilde{a}(\omega_0)|^2}{|r_{IM}(\omega_0)| \left|\frac{\partial r_r}{\partial \omega}\right|_{\omega_0}}$$

$$|\tilde{a}(\omega_0)|^2 = \left(r_{IM} \, T/m\right)\Big|_{\omega_0} \left|\frac{\partial r_r}{\partial \omega}\right|_{\omega_0}$$

noise — dissipn. — Temp — ω_0

→ Fluctuations set by { noise, [damping, collective modes] } response

i.e. $\omega \approx \omega_0$ natural frequency

→ $2\left(\frac{1}{2}kx^2\right) = 2\left(\frac{m\omega_0^2}{2}x^2\right) = T$

sets condition

Lesson: → Thermal equilibrium spectrum set by
- collective modes
- damping

resonances

- noise

→ F-D Thm links these, explicitly

→ For plasma, thermal equilibrium requires understanding
→ noise
→ collective modes
→ damping.